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Alternative New Notation for Quantum Information Theory

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The notation [Li, Song, and Luo (2002) Physics Letters A **297**, 121] has been generalized to quantum information theory. By this notation,we present three useful criteria for bipartite entanglement, a formula for general quantum swapping, and a formula for quantum teleportation of bipartite state by two EPR pairs.

KEY WORDS: quantum information; quantum entanglement; quantum computer.

1. INTRODUCTION

Since the concept of entanglement (Einstein *et al.*, 1935) was introduced into quantum physics, it plays a central roles in quantum information theory. Nearly all of quantum information processes are based on entangled state such as quantum dense code (Bennett and Wiesner, 1992), quantum swapping (Zukowski *et al.*, 1993), quantum teleportation (Bennett *et al.*, 1993) and so on.

In the quantum information theory some calculations are very complex. In quantum teleportation, we must write out all the joint states, and find out the relation between the *Alice*'s joint *Bell* states measurements and the receiver *Bob*'s local operation for his particle. If we perform a teleptortation of n-dimension state, the calculations are abjective. In another case, when we make a series of 2-dimensional quantum entanglement swapping, the calculations are very difficult. We have introduced a kind of notation (Li *et al.*, 2002), which was represented in Hughston *et al.* (1993), into quantum teleportation. In that paper, we gain a powerful theorem for general teleportation, by the notation.

In this paper we generalize the notation to much more quantum information process. By using this tool, we present three new criterias for bipartite entangled

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pure state, a uesful formula for quantum swapping and a creteria for bipartite state teleportation.

2. SOME DEFINITION

As we know, notation is often used in physics, which can simplify some calculation. Here we bring in following notation for quantum information in order to simplify some calculations and obtain some new conclusions.

In the paper (Li *et al.*, 2002), we introduced a notation for quantum teleportation, which is defined as follows,

Let $\mathbf{h}$ be a finite *Hilbert* space, and $\dim(\mathbf{h}) = N$, so 2-partite *Hilbert* space $\mathbf{H} = \mathbf{h} \otimes \mathbf{h}$.

We definite some useful notations :

Definition 1. $\alpha = (\alpha_0 \cdots \alpha_{N-2}\ \alpha_{N-1})$,

Definition 2. $\mathbf{e}_i = (\,|0\rangle_i \cdots |N-2\rangle_i\ |N-1\rangle_i\,)$,

Definition 3. $\tilde{\mathbf{e}}_i = (\,{}_i\langle 0| \cdots {}_i\langle N-2|\ {}_i\langle N-1|\,)$.

Let set $\{\,|0\rangle_i \cdots |N-2\rangle_i\ |N-1\rangle_i\,\}$ be a group of orthonormal complete basis of the i-th *Hilbert* space $\mathbf{h}_i$. So an arbitrary state $|\varphi\rangle = \sum_{i=0}^{N-1} \alpha_i\,|i\rangle$ can be written as $|\varphi\rangle = \alpha\mathbf{e}^t$ (where superscrip t means transposition). Arbitrary bipartite state $|\psi_{12}\rangle = \sum_{ij} a_{ij}\,|i\rangle_1\,|j\rangle_2$ can be written as $|\psi_{12}\rangle = \mathbf{e}_1\mathbf{A}\mathbf{e}_2^t = \mathbf{e}_2\mathbf{A}^t\mathbf{e}_1^t$, where $\mathbf{A}$ is a known matrix $\mathbf{A}_{ij} = a_{ij}$ and A^t is the transposed of A. If we perform a local operation U_1 on particle 1, the state become $U_1\,|\psi_{12}\rangle = \mathbf{e}_1 U_1\mathbf{A}\mathbf{e}_2^t$, and it is $U_2\,|\psi_{12}\rangle = \mathbf{e}_1\mathbf{A}U_2^t\mathbf{e}_2^t$ with local operation U_2.

We will show how this notation work in quantum information from the following discussion.

3. APPLICATION OF THE NOTATION FOR ENTANGLEMENT RESEARCH

The phenomenon of entanglement is a remarkable feature of quantum theory. It play a crucial role in the discussions of quantum mechanics and quantum information theory. In particular, the pure bipartite entangled state (PBES) is used in most quantum information transported process such as quantum teleportation, quantum code *et al.* So how to find out the PBES from PBS (pure bipartite state) is important. As we know, any pure bipartite entangled state can be expressed as **Schmidt** decomposition, so the criteria for pure bipartite entangled state is, if the **Schmidt** number of the pure bipartite state is more than one. by the notation, a pure bipartite state $|\psi_{12}\rangle$ can be described as

$$\begin{aligned}|\psi_{12}\rangle &= \sum_{ij} a_{ij}\,|i\rangle_1\,|j\rangle_2 \\ &= (|0\rangle_1\,|1\rangle_1\cdots|n-1\rangle_1)\,A\,(|0\rangle_2\,|1\rangle_2\cdots|n-1\rangle_2)^t \\ &= \mathbf{e}_1\mathbf{A}\mathbf{e}_2^t.\end{aligned}$$

Where all entangled information are included in the matrix A. Here we proposed a criteria for this.

Criteria 1: *The state $|\psi_{12}\rangle$ is entangled state, if and only if there are more than one non-zero eigenvalues of the matrix $AA^\dagger$, the number of the non-zero eigenvalues is just* **Schmidt** *number.*

Proof: As we know, any bipartite entangled state can be written as **Schmidt** decomposition, namely

$$|\psi_{12}\rangle = \sum_{ij} a_{ij}\,|i\rangle_1\,|j\rangle_2 = \sum_k \sqrt{P_k}|\widetilde{k}\rangle_1|\widetilde{k}\rangle_2$$

the joint density operator is

$$\rho\,(\psi_{12}) = \sum_{ijlm} a_{ij}\,(a_{lm})^\dagger\,|i\rangle_1\,|j\rangle_2\,\langle m|_2\,\langle l|_1$$

and the density operator of the qubit 1 is

$$\begin{aligned}\rho_1\,(\psi_{12}) &= \sum_{ilm} a_{im}\,(a_{lm})^\dagger\,|i\rangle_1\,\langle l|_1 \\ &= AA^\dagger = B\end{aligned}$$

We can calculate out the eigenvalues of $\rho_1\,(\psi_{12})$

$$\det(\lambda - B) = 0 \Longrightarrow \lambda$$

where $\lambda = (\lambda_1\lambda_2\cdots\lambda_n)$ and **Schmidt** number $N_s \geq 2$ then the state $|\psi_{12}\rangle$ is entangled state, namely there are more than two non-zero eigenvalue of the state $|\psi_{12}\rangle$.

$$\left|\widetilde{k}\right\rangle_1 = \sum_j \acute{t}_{kj}\,|j\rangle_1 \quad \text{and} \quad \rho_1\,(\psi_{12})\left|\widetilde{k}\right\rangle_1 = \lambda_k\left|\widetilde{k}\right\rangle_1$$

Let $|\psi_{12}\rangle = \sum_k\sqrt{\lambda_k}|\widetilde{k}\rangle_1|\widetilde{k}\rangle_2$, where $|\widetilde{k}\rangle_2 = \sum_i \alpha_i^k\,|i\rangle_2$ with ${}_2\langle\widetilde{k}|\widetilde{j}\rangle_2 = \delta_{kj}$, considered $|\psi_{12}\rangle = \sum_{ij} a_{ij}\,|i\rangle_1\,|j\rangle_2$, we can get all α_i^k. So $\left|\widetilde{k}\right\rangle_2$ is known. $|\psi_{12}\rangle = \sum_k \lambda_k|\widetilde{k}\rangle_1|\widetilde{k}\rangle_2$ □

From this criteria, we can get the **Schmidt** number of the state, and gain the **Schmidt** decomposition expression for the state.

For exmaple, we consider a state of 2-bit

$$|\psi\rangle_{12} = a\,|00\rangle_{12} + b\,|01\rangle_{12} + c\,|10\rangle_{12} + d\,|11\rangle_{12}\,,$$

if this state is entangled state. in order to simply we assume that all cofiecients is real number, we have

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix}.$$

the eigenequation of B is

$$\lambda^2 - tr\,(B)\,\lambda + \det(B) = 0$$

By the Newton's relations of the polynomial equation , we know if and only if $(ad - bc) \neq 0$,the above eqution have two non-zero real root,which means that state $|\psi\rangle_{12}$ is an entangled state.

Criteria 2: *The state* $|\psi_{12}\rangle$ *is entangled state, if and only if the rank of the matrix A is more than one, it is just the* **Schmidt** *number*.

Proof: As we know, the non-zero eigenvalue number of any matrix is equal to its rank.

There must be matrix T , $TT^\dagger = M_{ii}\delta_{ij}$ with $M_{ii} \in \{0, 1\}$

$$TAT^\dagger = K, \text{ with } K_{ij} = a_{ij}\delta_{ij}$$

and

$$TA^\dagger T^\dagger = K^\dagger, \text{ with } K_{ij}^\dagger = b_{ij}\delta_{ij}$$

so

$$T\rho_1 T^\dagger = TAT^\dagger TA^\dagger T^\dagger = KK^\dagger = \sigma_{ij}\delta_{ij}$$

with $\sigma_{kk} = a_{kk}b_{kk}$. finally we can easily get

$$R\,(\rho_1) = R\,(A) = R(A^\dagger),$$

where $R\,(A)$ is the rank of the matrix A. □

Definition 4. Entanglement of bipartite pure state is

$$E = \Pi_k \lambda_k$$

where λ_k is the k-th non-zero eigenvalue of the partitial density operator ρ_1, namely $\rho_1 = tr\,\{\rho_{12}\}_2$. Considering criteria 1 and definition 4, we can easily obtain new criteria as following.

Criteria 3: *The entanglement of the state* $|\psi_{12}\rangle$ *is*

$$E\,(\psi_{12}) = \Pi_k \lambda_k.$$

which equal to linear entropy of the state $|\psi_{12}\rangle$.

4. APPLICATION FOR QUANTUM SWAPPING AND BIPARTITE STATE QUANTUM TELEPORTATION

Quantum information attracted more attention since quantum teleportation has been proposed. Quantum swapping is important in quantum information. But there are some complicated calculation in quantum cryptography (Lee *et al.*, 2002) and quantum swapping, furthermore multi-dimension teleportation or series of swapping, while by this notation, all of this kinds of calculation become easy.

4.1. Quantum Swapping

As we know, any quantum information process is based on entangled state. If sender and receiver do not share any common entangled state, how they to communicate each other? The answer is that they can make a quantum swapping through another person as follow. *Alice* (sender) has a entangled states $|\phi_{12}\rangle$, *Bob*(receiver) has a entangled states $|\varphi_{34}\rangle$, *Alice* send qubit 2 to Charley(the third person) and *Bob* send qubit 3 to him, then Charley make a joint measurement on qubit 2 and 3, he told *Alice* and *Bob* his measured result, then *Alice* and *Bob* share a new entangled state $|\psi_{14}\rangle$, this process is quantum swapping.

$$|\phi_{12}\rangle = (|0\rangle_1\ |1\rangle_1 \cdots |n-1\rangle_1)\, A\, (|0\rangle_2\ |1\rangle_2 \cdots |n-1\rangle_2)^t$$
$$|\varphi_{34}\rangle = (|0\rangle_3\ |1\rangle_3 \cdots |n-1\rangle_3)\, C\, (|0\rangle_4\ |1\rangle_4 \cdots |n-1\rangle_4)^t$$

measured result

$$|\phi'_{23}\rangle = (|0\rangle_2\ |1\rangle_2 \cdots |n-1\rangle_2)\, B\, (|0\rangle_3\ |1\rangle_3 \cdots |n-1\rangle_3)^t$$

so the new entangled state

$$\begin{aligned} |\psi_{14}\rangle &= \langle\phi'_{23}|\,(|\phi_{12}\rangle \otimes |\varphi_{34}\rangle) \\ &= (|0\rangle_1\ |1\rangle_1 \cdots |n-1\rangle_1)\, A\overline{B}C\, (|0\rangle_4\ |1\rangle_4 \cdots |n-1\rangle_4)^t \\ &= \rho\,(|0\rangle_1\ |1\rangle_1 \cdots |n-1\rangle_1)\, F\, (|0\rangle_4\ |1\rangle_4 \cdots |n-1\rangle_4)^t \end{aligned}$$

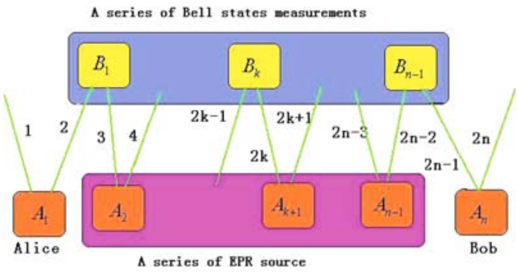

Fig. 1. Quantum swapping of n EPR pairs.

where

$$\begin{aligned}|\psi_{14}\rangle &= (|0\rangle_1\ |1\rangle_1 \cdots |n-1\rangle_1)\, F\, (|0\rangle_4\ |1\rangle_4 \cdots |n-1\rangle_4)^t \\ F &= A\overline{B}C/\rho(\overline{B}_{ij} = B^*_{ij}) \\ \rho &= \sqrt{tr\{(A\overline{B}C)(A\overline{B}C)^\dagger\}}.\end{aligned} \tag{1}$$

Further more, there are n bipartite entangled states $\{|\phi_{12}\rangle|\phi_{34}\rangle \cdots |\phi_{(2n-1)2n}\rangle\}$ and $|\phi_{(2k+1)(2k+2)}\rangle = \mathbf{e}_{2k-1}\mathbf{A}_k\mathbf{e}^t_{2k}\ |\phi'_{(2k)(2k+1)}\rangle = \mathbf{e}_{2k}\mathbf{B}_k\mathbf{e}^t_{2k+1}$, then through $n-1$ quantum swappings the final entangled state of particle 1 and particle $2n$ is $|\psi_{1(2n)}\rangle = \mathbf{e}_1\mathbf{F}\mathbf{e}^t_{2n}$,where

$$\mathbf{F} = \left(\prod_{k=1}^{n-1}\mathbf{A}_k\overline{\mathbf{B}}_k\right)\mathbf{A}_n. \tag{2}$$

In fact, quantum communication can be realized by sending one of the entanglement pairs, while during long way transformation the entanglement will decrease and some information will be lost. However we can overcome this problem by a series quantum swapping as shown in the Fig. 1 (where A_k means the k-th EPR pairs, and the B_k is the k-th joint measerment). The calculation of the series quantum swapping is very complex even n is a large number, but by the notation, the calculation become concise.

4.2. Bipartitie State Teleportation

Earlier studies have been confined to the teleportation of single-body quantum states (Bennett *et al.*, 1993), which is only a special case. Recently teleportation of multi-body quantum states was attracted more attentions (Lee and Kim 2002; Wang, 2001) the studies are only confined to teleportation of bipartite entangled state. In this paper we study on how to teleport a bipartite states by two EPR states. Based on the notation (Li *et al.*, 2002), we bring out how to construct Bob's local operation on his particles. Supposed that *Alice* was going to teleport a bipartite state

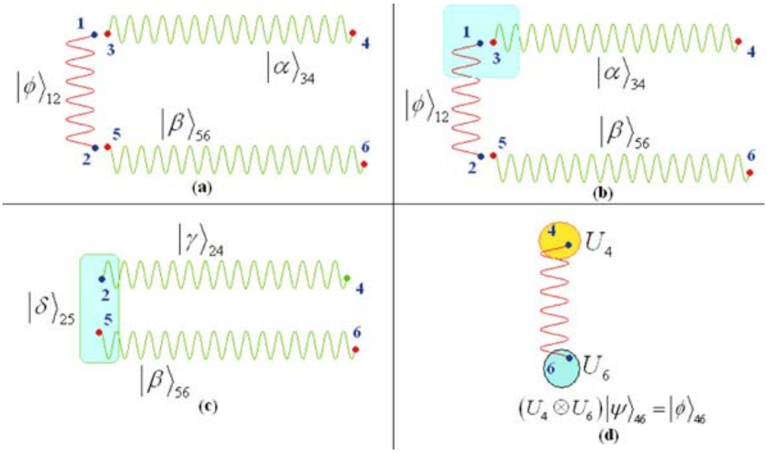

Fig. 2. Teleport bipartite state by using two bipartite entangled states.

$|\phi\rangle_{12} = \Sigma_{ij} A_{ij} \, |i\rangle_1 \, |j\rangle_2 = \overrightarrow{e}_1 A \overrightarrow{e}_2^{\,t}$ to *Bob*, she shared two known entanglement states $|\alpha\rangle_{34} = \overrightarrow{e}_3 B \overrightarrow{e}_4^{\,t}$ and $|\beta\rangle_{56} = \overrightarrow{e}_5 C \overrightarrow{e}_6^{\,t}$ with *Bob*. *Alice* performed a joint states measurement on particle 1 and particle 3, the result $|\delta\rangle_{13} = \overrightarrow{e}_1 D \overrightarrow{e}_3^{\,t}$, by the Eq. (1) we get the state of particle 2 and particle 4 $|\gamma\rangle_{24} = \overrightarrow{e}_2 M \overrightarrow{e}_4^{\,t} = \overrightarrow{e}_4 M^t \overrightarrow{e}_2^{\,t}$, where $M = A\overline{D}B\rho_1$. Then *Alice* made another joint states measurement on particle 2 and particle 5, the result $|\eta\rangle_{25} = \overrightarrow{e}_2 F \overrightarrow{e}_5^{\,t}$, by the Eq. (1) the state of particle 4 and particle 6 $|\varphi\rangle_{46} = \overrightarrow{e}_4 N \overrightarrow{e}_6^{\,t}$ was easily obtained,
where $N = M^t \overline{F} C \rho_2 = B^t \overline{D}^t A^t \overline{F} C \rho_1 \rho_2$.

$$|\varphi\rangle_{46} = |\varphi\rangle_{64} = \overrightarrow{e}_4 \frac{B^t \overline{D}^t A^t \overline{F} C}{\rho_1 \rho_2} \overrightarrow{e}_6^{\,t}$$

Alice told *Bob* her joint states measurements, *Bob* perform unitary operator U_4 and U_6 on his particle 4 and particle 6 respectively,

$$U_6 U_4 \, |\varphi\rangle_{64} = |\phi\rangle_{64} \tag{3}$$

where $U_4 = (\frac{\overline{D}B}{\rho_1})^{-1}$ and $U_6 = (\frac{\overline{F}C}{\rho_2})^{-1}$. Then *Bob* captured the state $|\phi\rangle$. We find that Bob's local operator U_4 and U_6 is independent with the state $|\phi\rangle_{12}$, illustrated as Fig. 2.

5. CONCLUSION

After introduced a new notation for the quantum information, we obtained a criteria for pure bipartite entangled state and a theorem for bipartites quantum teleportation by two EPR pairs. We constructed the operaters performed by the *Bob* on his qubit in order to obtain perfect replica of teleported states.

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